

## CHAPTER 8

# Quantitative Skills and Advanced Calculus Topics in AP Physics C: Mechanics

This chapter focuses on some of the quantitative skills that are important in your AP Physics C: Mechanics course. These are not all of the skills that you will learn, practice, and apply during the year, but these are the skills you will most likely encounter as part of your laboratory investigations or classroom experiences, and potentially on the AP Physics C Exam.

## Vector Products

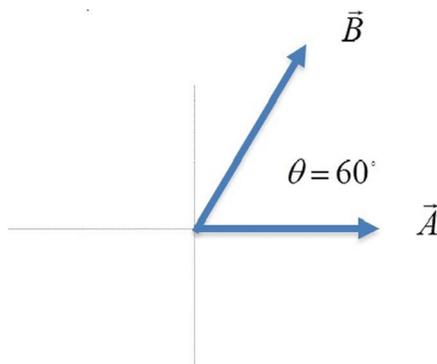
### Dot Product

The **dot product** is also called the scalar product. It is the product that takes two vectors and performs an operation where the product's value is a scalar value. Here is the definition of the dot product:

$$\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta$$

The angle theta is defined as the angle between the two vectors when the two vectors are drawn from a common origin.

Figure 8.1 shows the definition of the dot product:



**Figure 8.1:** Two Vectors (A and B) Drawn in x-y Coordinate Frame

### Example

Defining the two vectors in figure 8.1:

$$|\vec{A}| = 10 \text{ units}$$

$$|\vec{B}| = 12 \text{ units}$$

and also defining the vectors in unit vector notation:

$$\vec{A} = 10\hat{i}$$

$$\vec{B} = 6\hat{i} + 10.4\hat{j}$$

Using the definition of the dot product stated above, the product of vector A and vector B is

$$\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta = (10)(12) \cos 60^\circ = 60$$

This value of 60 is a scalar value (without direction!) and has the units' defined physical quantity.

There is a second way to compute the dot product. This process is defined by using the components of the vectors. Here is this definition:

$$\vec{A} \cdot \vec{B} = A_x \cdot B_x + A_y \cdot B_y + A_z \cdot B_z$$

Apply this rule to the example given, and we should get the same product result:

$$\vec{A} \cdot \vec{B} = (10)(6) + (0)(10.4) + (0)(0)$$

$$\vec{A} \cdot \vec{B} = 60$$

## Cross Product

The cross product is sometimes called the vector product. In this vector operation two vectors are taken through a process to produce a third vector. Here is one of the definitions of the cross product:

$$\vec{C} = \vec{A} \times \vec{B} = |\vec{A}| |\vec{B}| \sin \theta$$

The operation looks very similar to the dot product. The angle definition is also the same definition as the angle definition defined in the dot product. However, there is an important difference besides the obvious (the sine function contribution verses the cosine contribution). This difference is that there is a direction to the cross product. That direction is determined by the right-hand screw rule. The vector  $C$  is defined to be in the direction that is perpendicular to the plane created by the two vectors  $A$  and  $B$ . When rotating  $A$  into  $B$  with the right hand, the thumb will point in the direction of the vector created by the cross product. A good visual example of this is to take the cross product of the unit vector  $\hat{i}$  with the unit vector  $\hat{j}$ . So imagine rotating the  $x$ -axis into the  $y$ -axis with the right hand, and the thumb is in the  $z$ -direction. Therefore, the cross product vector is the unit vector  $\hat{k}$ . This is because the  $z$ -axis direction is perpendicular to the plane created by the  $x$ - $y$  axes. This is essentially what creates our right-handed coordinate systems.

### Example

Compute the cross product of the previously defined vectors ( $A$  and  $B$ ).

$$\vec{C} = \vec{A} \times \vec{B} = |\vec{A}||\vec{B}|\sin\theta = (10)(12)\sin 60 = 104$$

The vector  $C$  would have a magnitude of 104 units and a direction in the  $+z$  direction. The direction is determined by the right-hand rule, so the direction that is perpendicular to the plane contained by vectors in the  $x$ - $y$  plane is the  $+z$ -direction.

If you need more information, the following tutorials can help to further explain these concepts:



[Khan Academy: Vector dot product and vector length](#)

## Calculus for AP Physics C

### Derivatives

The derivative allows you to calculate the slope of a function or the rate of change in one quantity with respect to another quantity. For example, the slope of a position versus time graph would give us the velocity, so by taking the derivative of a position function, we can determine the velocity function.

### Power Rule

The power rule allows you to determine the derivative of a function of the form  $x(t) = t^n$  where  $n$  is a rational number. The power rule states:

$$x(t) = t^n$$

$$\frac{dx}{dt} = n \cdot t^{n-1}$$

If the position of a ball falling from rest from a height  $y_0$  is equal to  $y = y_0 - \frac{g}{2}t^2$ , the derivative of this function will give the velocity function.

$$\frac{dy}{dt} = v_y(t) = 0 - 2 \cdot \frac{g}{2}t^{(2-1)} = gt$$

$$v_y(t) = gt$$

### Example

Determine an expression for the velocity as a function of time for a ball with a position function equal to  $x(t) = x_0 + v_0t + \frac{1}{2}at^2$ .

$$\frac{dx}{dt} = v_x(t) = 0 + v_0 + \left(2 \cdot \frac{1}{2}at^1\right)$$

$$v_x(t) = v_0 + at$$

The following tutorial can help you learn more about the power rule:



[Khan Academy: Power rule](#)

## Chain Rule

For composite functions, the chain rule can help to find the derivative. For a function,  $y = f(g(x))$ , where  $y = f(h)$  and  $h = g(x)$ , the derivative of  $y$  with respect to  $x$  is

$$\frac{dy}{dx} = \frac{dy}{dh} \frac{dh}{dx} \quad \text{This could also be written as: } \frac{dy}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

A quick and easy way of remembering the chain rule is to think about the function  $y = f(g(x))$  as having an “inside function,”  $h = g(x)$ , and an “outside function,”  $y = f(h)$ . The derivative is then the derivative of the outside function (with the inside function left alone), times the derivative of the inside function.

### Example

Find the derivative of the function,  $y(x) = (5x - 8)^{\frac{1}{2}}$ .

In this case, the “outside function” is the  $\frac{1}{2}$  power (or square root) and the inside function is  $5x - 8$ . So the derivative  $\frac{dy}{dx}$  will be equal to:

$$\frac{dy}{dx} = \frac{1}{2}(5x - 8)^{\left(\frac{1}{2} - 1\right)} \cdot (5)$$

$$\frac{dy}{dx} = \frac{5}{2}(5x - 8)^{-\frac{1}{2}}$$

The following tutorial can help you learn more about the chain rule:



[Khan Academy: Chain rule](#)

## Antiderivatives and Definite Integrals

Evaluating a definite integral allows you to determine the area of the region between the graph of a function and the x-axis. That area gives you information about the net change in the quantity whose derivative is graphed.

Given a function,  $f(x)$ , the value of the definite integral would equal the difference between the values of the antiderivative at each of the two endpoints. For example, if  $f(x) = x^n$  then

$$\int_{x_1}^{x_2} f(x) dx = \int_{x_1}^{x_2} x^n dx = \frac{x^{(n+1)}}{(n+1)} \Big|_{x_1}^{x_2} = \frac{x_2^{(n+1)}}{(n+1)} - \frac{x_1^{(n+1)}}{(n+1)}$$

Note that the antiderivative for  $x^n$  inverts the operations in the power rule for derivatives.

Given an expression for velocity as a function of time for a car,  $v(t)$ , the net change in position for the car over an interval of time can be calculated by evaluating a definite integral of this function from  $t = 0$  to any time,  $t$ :  $\int_0^t v(t) dt$ . Given an initial position,  $x_0$ , the position,  $x$  as a function of time will be given by  $x(t) = x_0 + \int_0^t v(t) dt$ . For example, if  $v(t) = v_0 + at$ :

$$x(t) = x_0 + \int_{t_0=0}^t v(t) dt$$

$$x(t) = x_0 + \int_{t_0=0}^t (v_0 + at) dt$$

$$x(t) = x_0 + \left( v_0 t + \frac{at^2}{2} \right) \Big|_{t_0=0}^t$$

$$x(t) = x_0 + v_0 t + \frac{at^2}{2}, \text{ provided } t_0 = 0.$$

The following tutorial can help you learn more about anti-derivatives and integrals:



[Khan Academy: Anti-derivatives and indefinite integrals](#)

## First Order Differential Equations

A first order linear differential equation is a function that includes both a variable and the first derivative of that variable. For example, the function  $-x = 4 \frac{dx}{dt}$  is an example of a first order differential equation because it contains both  $x$  and the rate of change of  $x$  with respect to time  $\frac{dx}{dt}$ .

Steps to solving a simple first order differential equation:

1. Rearrange the equation so like variables are together (separate variables).
2. Integrate both sides, being sure to account for initial conditions in limits of integration.
3. Solve for an expression for the variable whose derivative is represented in the differential equation (in this case  $x$ ).

### Example

A box, given an initial speed  $v_0$ , slows to rest under the influence of air resistance, which applies a force  $F = -kv$ , where  $k$  is a constant and  $v$  is the velocity of the box as a function of time. Derive an expression for the velocity of the box as a function of time.

Since the force from the air is the only force slowing the box, the force from the air will be equal to the net force on the box.

$$\begin{aligned} \underbrace{\sum F}_{-kv} &= ma \\ -kv &= ma \\ -kv &= m \frac{dv}{dt} \end{aligned}$$

Now we can see that this is a differential equation since it contains both  $v$  and  $dv/dt$ . First rearrange the equation with like terms together:

$$\frac{-k}{m} dt = \frac{dv}{v}$$

Note that both  $k$  and  $m$  are constants, so  $k/m$  is a constant.

Now integrate both sides and add limits to account for initial conditions:

$$\begin{aligned} \int_{t=0}^t \frac{-k}{m} dt &= \int_{v_0}^v \frac{dv}{v} \\ \frac{-kt}{m} \Big|_{t=0}^t &= \ln v \Big|_{v_0}^v \\ \frac{-kt}{m} &= \ln v - \ln v_0 = \ln \left( \frac{v}{v_0} \right) \\ e^{\frac{-kt}{m}} &= e^{\ln \left( \frac{v}{v_0} \right)} \\ e^{\frac{-kt}{m}} &= \frac{v}{v_0} \end{aligned}$$

Finally, solve for  $v$ :

$$v_0 e^{\frac{-kt}{m}} = v(t)$$

The following tutorial can help you learn more about solving first order differential equations:



[Khan Academy: Separable equations introduction](#)

## Definition of Work (Calculus)

The complete definition of **work** involves two advanced mathematical ideas: the dot product and the integral. Here is the precise definition of work:

$$W = \int_A^B \vec{F} \cdot d\vec{r}$$

This would be defined as the integral of the dot product of the force and the displacement vector as the object is moved from position A to position B; “dr” is used to define a displacement linearly in any direction ( $x$ ,  $y$ ,  $z$ , or radial direction or combination). If the force being used in the definition is a constant force, then the definition can be written more simply (AP Physics 1 and 2 style) as

$$W = |\vec{F}| |\vec{r}| \cos\theta = F_{\parallel} \cdot d$$

where  $F_{\parallel}$  represents the parallel component of the force in the direction of the displaced distance  $d$ .

### Example

Let’s try a complete example using vectors and calculus. Assume a force has an expression of the following relationship:

$$F = 2x^2 \hat{i}$$

and the object has a displaced from a position in the  $x$ -direction of  $x = 0$  to  $x = 5$ .

The displacement vector is the position vector in the  $x$ - $y$  plane or  $d\vec{r} = dx \hat{i} + dy \hat{j}$ . This is what the computation would look like:

$$W = \int_{x=0}^{x=5} (2x^2 \hat{i}) \cdot (dx \hat{i} + dy \hat{j})$$

$$W = \int_{x=0}^{x=5} (2x^2 dx) = \left[ \frac{2}{3} x^3 \right]_{x=0}^{x=5} = \frac{250}{3} = 83.3 J$$

## Moment of Inertia

Rotation is a major concept in the AP Physics C: Mechanics course. One of the fundamental physical quantities in this area of mechanics is the idea of **rotational inertia**. This idea is qualitatively defined as the measurement of inertia of an extended body (system) in response to a torque acting on the system; in other words, how easy or difficult it is to mechanically rotate an extended body about some given rotational axis. The formal name for this type of inertia is the moment of inertia and the symbol for this quantity is  $I$ . The moment of inertia is mathematically defined in the following way for a discrete mass system:

$$I = \sum_i m_i r_i^2$$

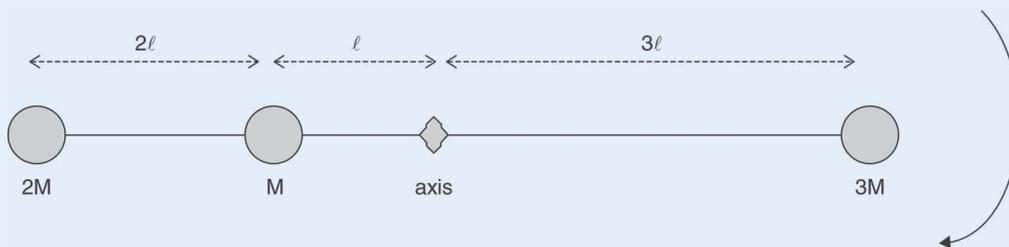
where  $I$  is inertia,  $m$  is the mass of the object, and  $r$  is the radius of rotation. The units for the moment of inertia measurement are  $\text{kg} \cdot \text{m}^2$ .

For a uniform shape or object (such as disks, spheres, rings, rods, etc.) there are known relationships for the moments of inertia. All of these relationships can be derived (although some of the derivations are beyond the scope of the AP Physics C course) from the calculus definition of the moment of inertia:

$$I = \int r^2 dm$$

### Example (noncalculus definition)

A very light (negligible mass) rigid rod is shown with three masses attached along the rod. Each mass has a value shown in the figure. The distances are also shown in terms of  $\ell$ . The system of masses will be rotated about the pivot point (center of rod) shown in the plane of the paper. The figure shows the rotating mass system viewed from above and the system is rotating in the plane of the paper. Compute the moment of inertia of this system.



The  $3M$  mass is located  $3\ell$  from the pivot, the  $M$  mass is located a distance of  $\ell$  from the pivot, and the  $2M$  mass is located a distance of  $3\ell$  from the pivot.

$$I = \sum m_i r_i^2$$

$$I = \left[ (3M \cdot (3\ell)^2) + (M \cdot (\ell)^2) + (2M \cdot (3\ell)^2) \right]$$

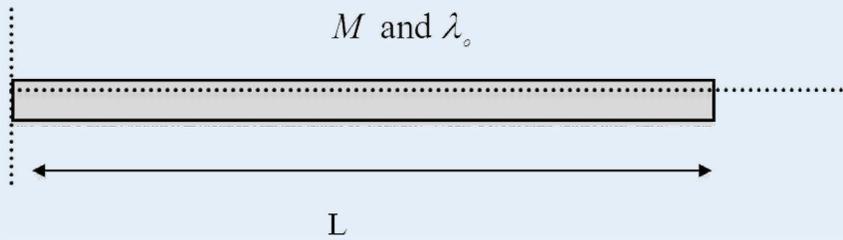
$$I = 46M\ell^2$$

### Example (calculus definition)

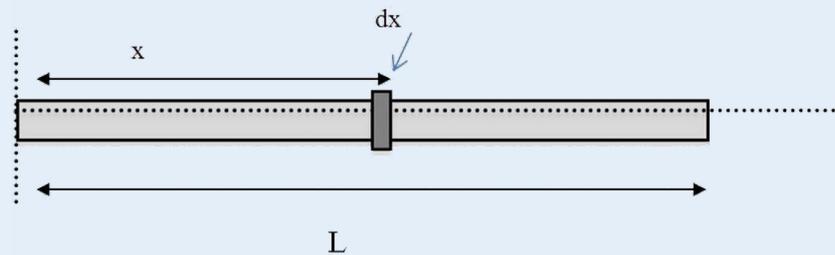
A rigid rod has a total mass of  $M$  and a length of  $L$ . The rod has a nonuniform linear mass density. The linear mass density is defined as

$$\lambda(x) = \lambda_0 \frac{x}{L}$$

This density varies with the position of  $x$  along the rod, where  $\lambda_0 = \frac{2M}{L}$  is a constant measured in  $\frac{\text{kg}}{\text{m}}$ . This means at greater distances of the  $x$ -position the density value will increase in direct proportion to the  $x$ -value: essentially the rod is weighted more on the far end. This is more of an example to show the mathematical nature of the moment of inertia definition and is not intended to be a realistic physical rod. Here is a diagram of our rod and frame of reference:



An infinitesimal slice mass along the rod needs to be defined as  $dm$ . The  $dm$  slice has an infinitesimal length in the  $x$ -direction that is defined as length of  $dx$ .



The calculus definition can now be applied to the rod. We will integrate the  $dm$  slices from a length of  $x = 0$  to a length of  $x = L$ .

$$dm = \lambda(x)dx = \lambda_0 \frac{x}{L} dx$$

$$r = x$$

$$I = \int dm x^2 = \int_0^L \lambda_0 \frac{x}{L} dx \cdot x^2 = \int_0^L \lambda_0 \frac{x^3}{L} dx = \left[ \lambda_0 \frac{x^4}{4L} \right]_0^L$$

$$I = \lambda_0 \frac{L^3}{4}$$

Let's give a value to  $\lambda_0$  of  $\frac{2M}{L}$  to further simplify the above expression for the moment of inertia of this particular rod, where  $M$  is the mass of the rod and  $L$  is the length of the rod. This reduces the expression of moment of inertia to a more familiar expression:

$$I = \lambda_0 \frac{L^3}{4} = \frac{2M}{L} \cdot \frac{L^3}{4} = \frac{1}{2} ML^2$$

If you need more information, the following tutorial can help to further explain these concepts:



[Khan Academy: More on moment of inertia](#)

# Total Mass Computation and Center of Mass Computation

## Total Mass Computation

Using the rod example from the last section, we can use calculus to determine that the total mass of the rod is  $M$ . This is more for a verification exercise, but it certainly gives more practice with integrating expressions in physics. To verify that the total mass of the rod has a value of  $M$ , we simply integrate the  $dm$  slices from zero to  $L$ .

$$dm = \lambda(x)dx = \lambda_0 \frac{x}{L} dx$$

$$M_{\text{total}} = \int_0^L dm = \int_0^L \lambda_0 \frac{x}{L} dx$$

$$M_{\text{total}} = \frac{\lambda_0}{L} \int_0^L x dx = \frac{\lambda_0}{L} \left[ \frac{x^2}{2} \right]_0^L = \frac{\lambda_0}{L} \cdot \frac{L^2}{2}$$

$$M_{\text{total}} = \lambda_0 \frac{L}{2} = \frac{2M}{L} \cdot \frac{L}{2} = M$$

## Center of Mass Computation

Using this same rod and variable linear mass density, we can use the calculus definition of the center of mass (CM) to determine the center of mass of this rod. The definition for the center of mass is

$$x_{\text{CM}} = \frac{\int x dm}{M_T}$$

### Example: center of mass using the calculus definition

We have already computed the total mass above of a long non uniform rod, (value is  $M$ ) and we simply do a different integral that is defined in the numerator of the CM expression.

$$\int_0^L x dm = \int_0^L x \cdot \lambda_0 \frac{x}{L} dx = \frac{\lambda_0}{L} \int_0^L x^2 dx = \frac{\lambda_0}{L} \cdot \left[ \frac{x^3}{3} \right]_0^L = \frac{\lambda_0 L^2}{3}$$

Then we substitute the value for  $\lambda_0$  into the expression:

$$\int_0^L x dm = \frac{\lambda_0 L^2}{3} = \frac{2M}{L} \cdot \frac{L^2}{3} = \frac{2}{3} ML$$

Lastly we divide this (numerator) by the total mass  $M$ , (the denominator in the CM expression) which gives:

$$X_{\text{cm}} = \frac{2}{3} L$$

So the CM of this nonuniformly massed rod is at a distance of two thirds along the rod and not the expected value of the CM being in the center of a uniformly weighted rod.

## Circular Motion and Rotation

### Definition of Torque in AP Physics C

**Torque** is used and defined in AP Physics 1. However, the definition that is used in AP Physics C is the definition without any modification or simplification. Torque is defined as

$$\vec{\tau} = \vec{r} \times \vec{F}$$

This can also be expressed in the same way as in AP Physics 1:

$$\tau = |\vec{r}| |\vec{F}| \sin \vartheta$$

where the magnitude of  $|\vec{r}| \sin \vartheta$  is defined as the moment arm.

Torque is a vector that comes from the vector cross product of two other vectors. The direction of torque is always a vector that is perpendicular to the plane that contains both of the vectors  $\vec{r}$  and  $\vec{F}$ . If  $\vec{r}$  and  $\vec{F}$  are also perpendicular to each other, then all three ( $\vec{r}$ ,  $\vec{F}$ ,  $\vec{\tau}$ ) vectors are mutually perpendicular to each other. Torque is measured in  $N \cdot m$ . The units are stated in this way to avoid confusion with the units of energy (joules).

If you need more information, the following tutorials can help to further explain these concepts:



[Khan Academy: Introduction to torque](#)

### Definition of Angular Momentum of a Linearly Moving Point Object

Because of the nature of the interaction with systems of point masses and extended bodies, it is sometimes necessary in physics to define a linearly moving object as having angular momentum about some conveniently (but mathematically arbitrary) taken frame of reference. This is commonly phrased as the angular momentum of an object about some point O.

The fact that a linearly moving body has an angular momentum does sound physically weird, but it is a necessary definition in mechanics. It is totally possible to define a point at which a linear moving particle has an angular momentum value about that point. For example, this becomes necessary when a baseball is thrown at an open door and has a collision. The momentum of the ball is transferred into angular momentum of the door as it rotates about the hinges. In order for our conservation laws to hold in all cases, we need a way to account for an angular momentum of the linearly moving ball. The definition for the angular momentum of a linear moving particle is

$$\vec{L} = \vec{r} \times \vec{p}$$

where  $\vec{r}$  is defined as the position vector. This vector is defined as being measured from the origin of the frame of reference defining the angular momentum to the position of the linearly moving object. The symbol  $\vec{p}$  is the linear momentum of the particle. The nature of this cross product gives rise to what is referred to as a moment arm in the world of rotation. The moment

arm is defined as the perpendicular distance between the line of the velocity vector and the line containing the axis of rotation. In other words, this distance (moment arm) is always the  $|\vec{r}|\sin\theta$  value from the cross product. Here is what this would look like in an example:

### Example: dart striking a wheel

Figure 8.2 shows a dart striking a wheel mounted on an axle free to rotate without any resistive forces. The dart clearly has linear velocity and momentum. The dart strikes the wheel and imbeds itself into the rubber wheel. The entire wheel/dart system begins to rotate about the central axle of the wheel with some constant angular velocity.

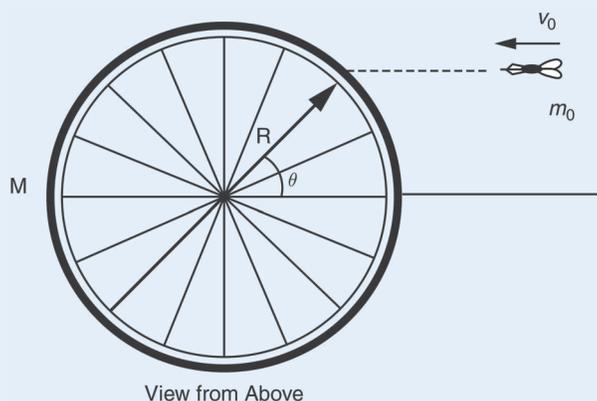


Figure 8.2: A Dart Striking a Wheel

In terms of  $M, m_0, v_0, R, \theta$  and fundamental constants as appropriate, determine the angular velocity of the wheel after the collision with the dart.

To determine the angular velocity of the wheel, the conservation of angular momentum must be used. There are no external torques on the system of the wheel and the dart. The only torque on the wheel or dart is exerted as the dart collides with the wheel, which would be an internal torque. This means that the total angular momentum of the system before the collision must be equal to the total angular momentum after the collision. Now we see why we need our definition! Since the wheel is at rest prior to the collision, it initially has no angular momentum. The only angular momentum is defined as the dart's angular momentum about the center of the wheel. It is the wheel and the final angular momentum that defines our point of reference for the entire approach to the problem.

$$\vec{L}_{\text{dart}} = \vec{r} \times \vec{p} = (R)(m_0)v_0 \sin\theta = mv_0R \sin\theta$$

After the collision the wheel and dart system is rotating and has an angular momentum equivalent to

$$L = I\omega$$

Note that the total moment of inertia of the system is the sum of the two inertias about the center of the wheel: the inertia of the wheel and the inertia of the dart stuck in the wheel a distance  $R$  away from the center of the wheel. Using the definition of moment of inertia of a ring (wheel) and the definition of a point mass moment of inertia, the total moment of inertia can be found.

$$I_{\text{system}} = I_{\text{wheel}} + I_{\text{dart}}$$

$$I_{\text{system}} = MR^2 + m_0R^2$$

This gives an angular momentum of the wheel after the collision as

$$L_{\text{wheel/system}} = I_{\text{system}}\omega = (M + m_0)R^2\omega$$

Now equate the angular momentum of the dart before the collision equal to the angular momentum of the wheel after the collision:

$$m_0v_0R\sin\theta = (M + m_0)R^2\omega$$

$$\omega = \frac{m_0}{M + m_0} \cdot \frac{v_0}{R} \sin\theta$$

This example shows why the definition of angular momentum for a linearly moving object is a necessary definition in mechanics.

If you need more information, the following tutorial can help to further explain this concept:



[Khan Academy: Angular momentum](#)